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THE ASSESSMENT OF MAJOR HAZARDS: ESTIMATION OF INJURY AND DAMAGE AROUND A HAZARD SOURCE USING AN IMPACT MODEL BASED ON INVERSE SQUARE LAW AND PROBIT RELATIONS

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Summary

An analytical model is described for the impact of a hazard on the surrounding area The basis of the model is a uniform population density, the inverse square law for the decay of the intensity of the physical effect and the lognormal distribution, or probit equation, for the relation between the causative, or injury, factor and the probability of injury It is shown that if these assumptions hold, the number of people injured may be approximately estimated by calculating the radius for 50% injury and assuming that all persons inside the circle suffer injury while all those outside it escape injury The error in this assumption is given by the factor $\phi = \exp(\sigma^2/2)$, where σ is the spread parameter of the lognormal distribution

Introduction

In a hazard assessment which is taken to the point of determining the risk to the public it is necessary to estimate injury in the area around the hazard source In order to do this it is necessary to be able to estimate first the intensity of the physical effect such as heat radiation, explosion overpressure or toxic dosage as a function of distance, and then the probability of injury as a function of this physical intensity

The paper describes an analytical model for the impact of a hazard on the surrounding area The model is based on the inverse square law for the decay of the intensity of the physical effect and on the lognormal distribution, and hence probit equations, for the relations between the causative, or injury, factor, which is a function of the intensity of the effect, and the probability of injury

A comparison is made between this model and an approximate model which is sometimes used In this latter the radius at which there is a 50%

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probability of injury is determined and it is then assumed that all persons within the circle suffer injury while all those outside suffer none This short cut method greatly reduces the amount of work required in an assessment It turns out that this model is quite accurate in some cases but grossly in error in others and that the crucial factor is the variance of the distribution describing the injury

Another significant result from the model is that the importance of any error in the probit equation depends on the variance of the distribution If the variance is small, some degree of error in it, and in the corresponding probit equation parameters, gives only a small error in the estimate of the number of people injured

For convenience the argument is based primarily on injury, but the same considerations apply to damage

The model depends on the applicability of the inverse square law to the intensity of the physical effect from the hazard source This is illustrated using simple models of thermal radiation, explosion overpressure and toxic concentration and dosage The extension to more complex models for these phenomena is not attempted

Approximate model

The approximate model, or short cut method, is to estimate the number of people who suffer injury from the relation

$$N_i = \pi r_{50}^2 d_p \tag{1}$$

where d_p is the population density (persons/m^2), N_i the total number of people injured and r_{50} the radius at which the probability, P, of injury is 0 5 (m)

In other words, it is assumed that the number of people within the circle of radius r_{50} who escape injury is approximately equal to the number of those outside the circle who suffer injury These two groups, therefore, cancel each other out and the net effect is as if all those within the circle but none of those outside suffer injury

If the probability of injury varies in a manner such as that shown in Fig 1, the assumption appears reasonable as applied to a line of persons exposed, since the two shaded areas of the figure will approximately cancel out If the application is to persons exposed over an area, however, the numbers in the outer annulus are likely to exceed those in the inner annulus At first sight, therefore, it appears that this approach must involve some degree of error

This approximate model is often suggested as a short cut method of estimation

An analogous approximation was tested in the Rijnmond Report [1] It was assumed in the toxic gas release model that all those inside and none of those outside the 50% lethal contour would be killed A check on this assumption showed that it was not very satisfactory for offsite deaths, but gave reasonable results for onsite deaths, where the cloud edge was relatively sharp

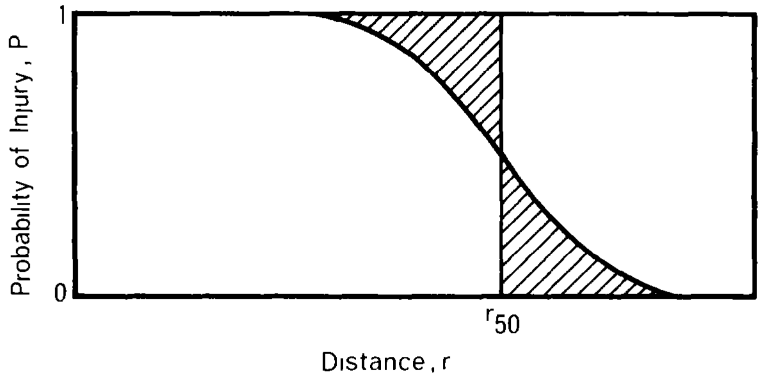

Fig 1 General form of the variation of probability of injury with distance

Injury—distance relations

In principle the intensity—distance, or decay, relations which describe the variation of the intensity of the physical effect with distance, depend on the particular physical phenomenon involved

In considering the intensity of the physical effect from a hazard source, it is convenient to define a radius r_o within which the intensity is so high that the probability of injury is unity The intensity is used here in the form of the normalised intensity, ι, which has the values

$$\iota > 1 \qquad r < r_o \qquad (2a)$$

$$\iota = 1 \qquad r = r_o \qquad (2b)$$

$$\iota < 1 \qquad r > r_o \qquad (2c)$$

The normalised intensity, ι, is obtained by dividing the actual intensity by the value of the intensity at which the probability of injury is effectively unity It is then a matter of indifference whether for values of r less than r_o the value used for ι in the relation for the estimation of the probability of injury is 1 or greater than 1

One common type of decay relation is exponential This may be represented by the relation

$$\iota = \exp[-(r - r_o)/r_s] \qquad r \geqslant r_o \qquad (3)$$

where r_s is a scaling parameter (m)

Another type of decay relation is the inverse square law This may be represented by the relations

$$\iota = \frac{1}{(r/r_o)^2} \qquad r \geqslant r_o \qquad (4a)$$

$$= (r_o/r)^2 \qquad (4b)$$

The radius r_0 may be regarded as broadly equivalent to the radius of the physical phenomenon considered. For example, it might be the radius of a fireball. However, in eqns (3) and (4) it should be treated simply as a parameter which is used to fit the calculated intensity decay curve.

In practice, many of the physical phenomena of importance in hazard assessment have an approximately inverse square law decay. The inverse square law applies to heat radiation from fires and fireballs. It applies approximately to peak overpressure from condensed phase explosions over the overpressure range of main interest. It also applies approximately to concentration and dosage of toxic gas in many situations, although here the position is more complex.

A fuller account of the applicability of the inverse square law to these different hazards is given in Appendix 1.

An inverse square law relation for the decay of the intensity of the physical effect is assumed in the treatment which follows.

Directional effects

Some physical phenomena are directional and affect not a circular area around the hazard source but an area which is a sector with its apex at the source. The most important phenomena of this type are those involving toxic gas release.

The model given here is fully applicable to this case, provided only this sector is considered and the sector is subdivided into a sufficient number of subsectors that the conditions along any arc in the subsector are essentially uniform.

Injury factor

The factor which correlates with injury is not necessarily the intensity of the physical effect, but may be some function of this, such as a power function or a time integral. For example, for eardrum rupture the injury factor is the peak overpressure itself, but for toxic deaths it is a time integral with concentration or time raised to some power. It is therefore this causative factor, or injury factor, which is used in injury correlations. This factor is used here in the form of the normalised injury factor, x.

Lognormal distribution and probit equation

The relation between the injury factor and the probability of injury tends to follow a lognormal distribution. For this distribution the density function, $f(x)$, is

$$f(x) = \frac{1}{(2\pi)^{1/2}\sigma x} \exp[-(\ln x - m^*)^2/2\sigma^2] \quad (5)$$

where m^* and σ are the location and spread parameters of the distribution

Where this is so the relation can conveniently be cast in the form of a probit equation The probit, Y, is defined by the relation

$$P = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{Y-5} \exp(-u^2/2)\, du \tag{6a}$$

$$= f(Y) \tag{6b}$$

The probability, P, is identical to the distribution function, $F(x)$, of the lognormal distribution Then

$$F(x) = \int_0^x f(x)\, dx \tag{7}$$

$$= P \tag{8}$$

The probit equation is usually written in the form

$$Y = k_1 + k_2 \ln x \tag{9}$$

Then from eqns (5)—(9)

$$k_1 = 5 - m^*/\sigma \tag{10a}$$

$$k_2 = 1/\sigma \tag{10b}$$

Probit equations are described by Finney [2] In particular, he discusses the relation between the lognormal distribution and the probit equation and gives tables and graphs relating probits and probabilities

Alternative model

The model of hazard impact now described was originally developed in order to investigate the degree of error in the approximate model The principal assumptions are that the intensity—distance relation for the physical effect is the inverse square law, that the injury factor—injury probability relation is the lognormal distribution and that the population is uniformly distributed around the hazard source from a radius of zero to a radius at which the effect becomes negligible. The assumption is also made that where the injury factor is not actually identical to the intensity of the physical effect it is a linear or at least a power law function of it. It can be seen from eqns (9) and (10) that this latter case is equivalent to a simple modification of the constants in the probit equation, or, alternatively, of the parameters of the lognormal distribution

The model consists of the following equations

$$N_1 = \int_0^{\infty} 2\pi d_p P(r) r\, dr \tag{11}$$

$$P = \frac{1}{(2\pi)^{1/2}\sigma} \int_0^x \frac{1}{x} \exp\left[-(\ln x - m^*)^2/2\sigma^2\right] dx \quad (12)$$

$$x = (r_o/r)^2 \quad (13)$$

It can be shown that eqns (11)—(13) yield the result

$$N_1 = \pi r_{50}^2 d_p \exp(\sigma^2/2) \quad (14)$$

The derivation of this equation is given in Appendix 2

Equation (14) may be written in the form

$$N_1 = \pi r_{50}^2 d_p \phi \quad (15)$$

with

$$\phi = \exp(\sigma^2/2) \quad (16)$$

where ϕ is a correction factor which allows for the effect of the variance Then comparing eqn (1) with eqns (15) and (16), it can be seen that eqn (15) reduces to eqn (1) as σ tends to zero and ϕ to unity

The characteristics of this model have been investigated numerically The cases considered, Cases 1 and 2, are shown in Table 1, and some results for these two cases are given in Figs 2(a) and 2(b), respectively In both cases the value of r_o is 100 m Case 1 is that of a distribution with a narrow spread (σ = 0 25), while Case 2 is that of a distribution with a wide spread (σ = 1)

TABLE 1

Estimation of injuries around a hazard source

	Case 1	Case 2
Parameters		
d_p (persons/m²)	0 001	0 001
r_o (m)	100	100
m^*	−1 83	−1 83
σ	0 25	1 00
Results		
r_{50} (m)	250	250
N_1 (eqn 1)	196	196
ϕ	1 03	1 65
N_1 (eqn 15)	202	318

The location parameter, m^*, is the same in the two cases (−1 83), and since it is this which determines the value of r_{50}, this also is the same at 250 m in both cases The figures show the variation with distance of the probability of injury and the number of people injured The probability of injury falls off with distance, while the number of people injured passes through a maxi-

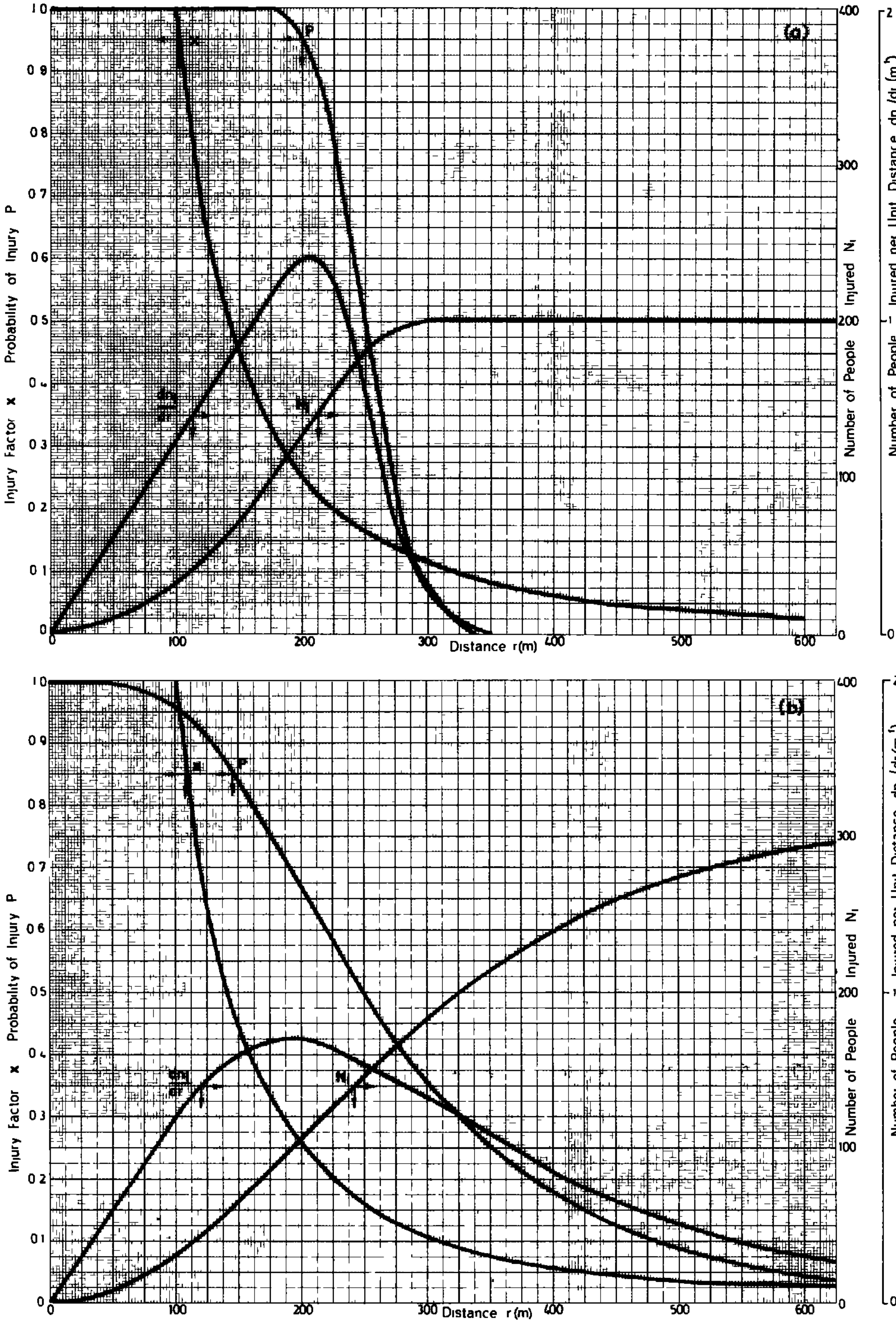

Fig 2 Probability of injury and number of people injured with distance for some generalised probit equations (a) Case 1, (b) Case 2

TABLE 2

Parameters of some probit equations and of the corresponding lognormal distributions

	Causative factor[a]	Probit equation parameters		Lognormal distribution parameters		Value of causative factor for $P = 0\ 999$	Parameter for normalised relations[b], m^*
		k_1	k_2	m^*	σ		
A Eisenberg equations [3]							
Fire							
Burn deaths from flash fire	$t_e I_e^{4/3}/10^4$	−14 9	2 56	7 77	0 391	7946	−1 21
Burn deaths from pool fire	$tI^{4/3}/10^4$	−14 9	2 56	7 77	0 391	7946	−1 21
Explosion							
Deaths from lung haemorrhage	p^o	−77 1	6 91	11 9	0 145	226049	−0 447
Eardrum ruptures	p^o	−15 6	1 93	10 7	0 518	214188	−1 60
Deaths from impact	J	−46 1	4 82	10 6	0 207	76323	−0 641
Injuries from impact	J	−39 1	4 45	9 91	0 225	40315	−0 694
Injuries from flying fragments	J	−27 1	4 26	7 54	0 235	3868	−0 725
Structural damage	p^o	−23 8	2 92	9 86	0 342	55339	−1 06
Glass breakage	p^o	−18 1	2 79	8 28	0 358	11933	−1 11
Toxic release							
Chlorine deaths	$\Sigma C^{2\ 75}T$	−17 1	1 69	13 1	0 592	2973723	−1 83
Chlorine injuries	C	−2 40	2 90	2 55	0 345	37 23	−1 07
Ammonia deaths	$\Sigma C^{2\ 75}T$	−30 57	1 385	25 7	0 722	$1\ 326 \times 10^{12}$	−2 23

B Other equations					
Canvey First Report [5]					
Ammonia deaths					
Equation a	$\Sigma C^{2\ 75}T$	1 14	0 782	4 94	1 28
Equation b	$\Sigma C^{2\ 75}T$	−7 41	2 205	5 63	0 454
Rijnmond Report [1]					
Hydrogen sulphide deaths	$\Sigma C^{2\ 5}T$	−41 48	2 366	19 6	0 423
Acrylonitrile deaths	ΣCT	−33 39	3 981	9 64	0 251
ten Berge and van Heemst [6]					
Ammonia deaths	$\Sigma C^{2}T$	−35 7	1 90	21 4	0 526
Chlorine deaths	$\Sigma C^{2}T$	−6 5	0 5	23 0	2 00
Harris and Moses [7]					
Chlorine deaths	$\Sigma C^{2\ 75}T$	−11 4	0 82	20 0	1 22

[a] C is concentration (ppm), except for Canvey First Report (g/m^3), and ten Berge and van Heemst (mg/m^3), I is radiation intensity (W/m^2), J is impulse (N s/m^2), p^o is peak overpressure (N/m^2), t is time (s), T is time interval (min) Subscript e is effective

[b] The value of k_1 is 8 09 in all cases

mum For Case 1 the estimates of the number of people injured are 196 using the approximate equation (eqn 1) and 202 using the exact equation (eqn 15), while for Case 2 the corresponding values are 196 and 318 The error in the approximate model is therefore much greater for Case 2, that with the higher variance

Probit equations for industrial hazards

A number of probit equations applicable to industrial hazards have been given in the literature The largest set appears of be that of Eisenberg et al [3], which cover fire, explosion and toxic hazards and which have been summarised by Lees [4] These probit equations were stated by these authors to be very approximate and they have been the subject of some criticism Other authors [5—7] have given further probit equations, which are mainly for toxic hazards and some of which represent revisions of the Eisenberg equations

A selection of these probit equations is given in Table 2 The table gives the parameters of the probit equations and of the corresponding lognormal distributions for the case where the injury factor is unnormalised Alternatively, the injury factor may be normalised ($x \leqslant 1$) It is convenient to normalise by the value of the injury factor which corresponds to a probability of injury of 0 999 Table 2 gives this value of the injury factor and the parameters of the probit equations and of the corresponding lognormal distribution for the normalised case for some of the hazards

Discussion

From the model derived for the common case of inverse square law decay the number of people injured is given by eqn (15) This equation reduces to the approximate model of eqn (1) as the spread parameter, σ, tends to zero and the variance correction factor, ϕ, tends to unity

The values of σ and ϕ for the probit equations given in Table 2 lie in the following ranges

No of equations		
12	$\sigma < 0\ 6$	$1 < \phi < 1\ 2$
3	$0\ 6 < \sigma < 0\ 9$	$1\ 2 < \phi < 1\ 5$
2	$\sigma > 0\ 9$	$\phi > 1\ 5$

The value of the spread parameter, σ, depends on the degree of homogeneity within the overall population Lack of homogeneity may be due to differences in susceptibility within a nominally similar population, e g , healthy adults, and in part due to mixing of populations, e g , healthy adults and children and/or old people

The smallness of the variance correction factor for the case where the value of the spread parameter is low has important implications In deriving injury—intensity relations there is generally greater confidence in the 50% injury value than in other values such as, say, those for 10% or 90% injury This in turn means that there is greater confidence in the value of m^* than there is in that of σ The likely form of the limits on the relation between the injury factor and the probit for injury is sketched in Fig 3, which shows that the limits tend to be relatively close at the 50% injury value but to diverge for lower and higher values The results obtained here indicate that, provided σ is small, the estimate of the number of people injured is not very sensitive to errors in the value of this parameter

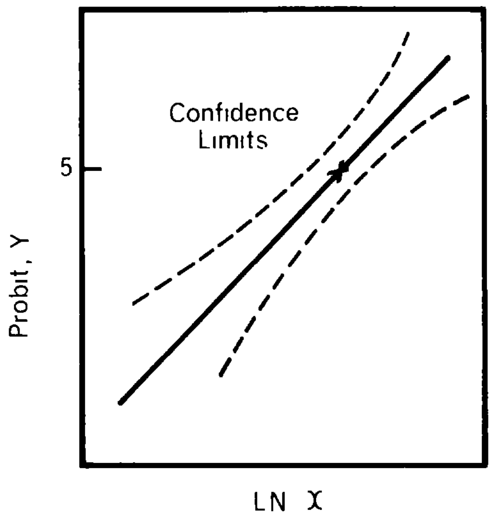

Fig 3 General form of the confidence limits for relation between injury factor and probit for injury

It follows from this that in using this model to make an assessment or to determine the error involved it may be preferable to use separate probit equations for homogeneous sections of the population, each with a relatively small spread, rather than a single probit equation for the whole population with a large spread

The value of the model is seen, however, as much in increasing understanding of the impact of hazards on the surrounding area as in providing a short cut estimation method

The model is based on the assumption of the inverse square law for the decay of the intensity of the physical effect This has been justified using simple models for fire, explosion and toxic release Another assumption is that the injury factor is related to the intensity of the physical effect linearly or by a power law These two features would bear further study

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List of symbols

C	concentration (various units — see text) (Table 2 only)
C	diffusion parameter ($m^{n/2}$) (Appendix 1 only)
d_p	density of population (persons/m^2)
D_{tid}	total integrated dosage ((kg/m^3) min)
E	heat radiated by fireball (kW)
$f(x)$	density function for injury
F	heat radiated by fireball on target (kW/m^2) (Appendix 1 only)
$F(x)$	distribution function for injury
i	normalised intensity of physical effect
J	impulse (N s/m^2)
k_1, k_2	constants in probit equation
L	distance from centre of fireball to target (m)
m	alternative location parameter in lognormal distribution
m^*	location parameter in lognormal distribution
n	diffusion index
$n_i(r)$	number of people injured at distance r
N_i	total number of people injured
p^o	peak overpressure of explosion (N/m^2)
P	probability of injury
Q	mass rate of release (kg/s)
Q^*	mass released (kg)
r	radial distance (m)
r_o	radius of physical phenomenon (m)
r_s	scaling parameter (m)
t	time (various units — see text)
T	time interval (min)
u	wind velocity (m/s)
W	mass of explosive (kg)
x	normalised injury factor
x	downwind distance (m) (Appendix 1 only)
Y	probit
z	scaled distance (m/$kg^{1/3}$)
σ	spread parameter in lognormal distribution (σ^2 = variance)
χ	concentration (kg/m^3)
ϕ	correction factor for effect of variance
Φ	normal distribution function

Subscript

50	for probability of injury equal to 0.5

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Appendix 1

The simpler models for the physical phenomena which are of interest in hazards work tend to give an approximately inverse square law for the decay of the intensity of the physical effect with distance

The heat radiation from a fireball is given by Roberts [8] as

$$F = E/4\pi L^2 \qquad \text{(A1 1)}$$

where E is the heat radiated (kW), F the heat radiated on the target (kW/m^2) and L the distance from the centre of the fireball to the target (m)

Similar equations apply to other types of fire such as pool fires and flares Thus the equation for heat radiation from a flare given by Hajeck and Ludwig [9] and quoted in API RP 521 [10] has this general form

The peak overpressure from the explosion of a high explosive is a function of the scaled distance

$$p^{o} = f(z) \qquad \text{(A1 2)}$$

with

$$z = r/W^{1/3} \qquad \text{(A1 3)}$$

where p^{o} is the peak incident overpressure (N/m^2), W the mass of explosive (kg) and z the scaled distance ($m/kg^{1/3}$) The function in eqn (A1 2) is usually provided in graphical form, such as the curves given by Baker et al [11] However, over a large part of the overpressure range of practical interest the curve, which is a log—log plot, has a slope of approximately −2 so that

$$p^{o} \propto 1/r^2 \qquad \text{(A1 4)}$$

The variation with distance of the concentration of a toxic gas cloud is more complex Here again, however, many situations approximate to an inverse square law

For a neutral density gas a commonly used set of equations are those of Sutton [12] For an instantaneous release the concentration χ (kg/m^3) at ground level on the centre line of the cloud and at cloud centre is

$$\chi = \frac{2Q^*}{\pi^{3/2} C^3 (ut)^{(3/2)(2-n)}} \qquad \text{(A1 5a)}$$

For neutral conditions, $n = 0.25$

$$\chi \propto 1/x^{2.6} \qquad \text{(A1 5b)}$$

The total integrated dosage, D_{tid} ((kg/m^3) s), is

$$D_{\text{tid}} = \frac{2Q^*}{\pi C^2 u (ut)^{2-n}} \qquad \text{(A1 6a)}$$

Setting $n = 0.25$

$$D_{\text{tid}} \propto 1/x^{1.75} \qquad \text{(A1 6b)}$$

For a continuous release the concentration at ground level on the centre line of the cloud is

$$\chi = \frac{2Q}{\pi C^2 u x^{2-n}} \qquad \text{(A1 7a)}$$

Setting $n = 0.25$

$$\chi \propto 1/x^{1.75} \qquad \text{(A1 7b)}$$

The total integrated dosage is obtained by assuming that the continuous release in fact lasts for some finite time Then the total integrated dosage is

$$D_{\text{tid}} \propto 1/x^{1.75} \qquad \text{(A1 8)}$$

As stated earlier, the decay predicted by more complex models, such as those for unconfined vapour cloud explosion and heavy gas dispersion, is beyond the scope of the present treatment

Appendix 2

The model considered is that in eqns (11)—(13) An alternative location parameter, m, may be defined

$$m = \exp m^* \quad \text{(A2 1a)}$$

$$m^* = \ln m \quad \text{(A2 1b)}$$

Some other relations which are relevant are as follows The injury factor x_{50} and the radius r_{50} at which the probability of injury is 50% ($P = 0\ 5$) are obtained from eqns (9)—(10) by putting $Y = 5$ so that

$$x_{50} = \exp m^* \quad \text{(A2 2a)}$$

$$= m \quad \text{(A2 2b)}$$

From eqn (13)

$$r_{50} = r_o / x_{50}^{1/2} \quad \text{(A2 3)}$$

$$= r_o / m^{1/2} \quad \text{(A2 4a)}$$

$$= r_o / \exp(m^*/2) \quad \text{(A2 4b)}$$

It may be noted that as m^* tends to zero, r_{50} tends to r_o

Now let

$$y = \ln x \quad \text{(A2 5)}$$

Then

$$P(r) = \frac{1}{(2\pi)^{1/2}\sigma} \int_{-\infty}^{y} \exp\left[-(y - m^*)^2/2\sigma^2\right] \mathrm{d}y \quad \text{(A2 6)}$$

Let

$$z = (y - m^*)/\sigma \quad \text{(A2 7)}$$

Then

$$P(r) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{z} \exp(-z^2/2)\, \mathrm{d}z \quad \text{(A2 8a)}$$

$$= \Phi(z) \quad \text{(A2 8b)}$$

where $\Phi(z)$ is the normal cumulative distribution function

Then, from eqns (13), (A2 5), (A2 7) and (A2 8), eqn (11) becomes

$$N_1 = \int_{0}^{\infty} 2\pi d_p \Phi\left[(2\ln r_o - 2\ln r - m^*)/\sigma\right] r\, \mathrm{d}r \quad \text{(A2 9)}$$

Let

$$u = (2\ln r_o - 2\ln r - m^*)/\sigma \qquad \text{(A2 10)}$$

Then

$$N_1 = \pi r_o^2 d_p \exp(-m^*) \int_{-\infty}^{\infty} \sigma \exp(-\sigma u)\Phi(u)du \qquad \text{(A2 11a)}$$

$$= \pi r_o^2 d_p \exp(-m^*) I \qquad \text{(A2 11b)}$$

where

$$I = \int_{-\infty}^{\infty} \sigma \exp(-\sigma u)\Phi(u)du \qquad \text{(A2 12)}$$

Integrating by parts

$$I = [-\exp(-\sigma u)\Phi(u)]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \exp(-\sigma u)\Phi'(u)\, du \qquad \text{(A2 13)}$$

The limits are

$u \to \infty \quad \exp(-\sigma u)\Phi(u) = 0 \times 1 = 0$

$u \to -\infty \quad \exp(-\sigma u)\Phi(u) = 0$

The derivation of the second of these limits is as follows For $u < -4\sigma$

$$\Phi(u) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{u} \exp(-v^2/2)\, dv < \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{u} \exp(2\sigma v)\, dv =$$

$$\frac{1}{(2\pi)^{1/2} 2\sigma} \exp(2\sigma u) \qquad \text{(A2 14)}$$

since for $v < -4\sigma$, $\exp(-v^2/2) < \exp(2\sigma v)$ Hence for sufficiently large negative u, $\exp(-\sigma u)\Phi(u) < k \exp(\sigma u)$ Hence the first term in eqn (A2 13) is zero

Then

$$I = \int_{-\infty}^{\infty} \exp(-\sigma u)\Phi'(u)\, du \qquad \text{(A2 15a)}$$

$$= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \exp(-\sigma u) \exp(-u^2/2)\, du \qquad \text{(A2 15b)}$$

Completing the square in the combined exponent

$$I = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \exp(\sigma^2/2) \exp[-(u + \sigma)^2/2] \, du \qquad (A2\ 16)$$

$$= \exp(\sigma^2/2) \qquad (A2\ 17)$$

since the remaining factor is simply the normal probability integral with mean $-\sigma$ and variance 1

Then eqn (A2 11) becomes

$$N_i = \pi r_0^2 d_p \exp(-m^*) \exp(\sigma^2/2) \qquad (A2\ 18)$$

and hence from eqn (A2 3)

$$N_i = \pi r_{50}^2 d_p \exp(\sigma^2/2) \qquad (A2\ 19)$$

$$= \pi r_{50}^2 d_p \phi \qquad (A2\ 20)$$

where

$$\phi = \exp(\sigma^2/2) \qquad (A2\ 21)$$